Chapter 3

BASICS OF OPTIMIZATION

3.1 INTRODUCTION

Ingredients of Optimization

- Decision variables ($x \in \mathbf{R}^n$): undetermined parameters
- Cost function ($f : \mathbf{R}^n \to \mathbf{R}$): the measure of preference
- Constraints ($h(x) = 0$, $g(x) \leq 0$): equalities and inequalities that the decision variables must satisfy

$$\min_{x \in \mathbf{R}^n} f(x)$$

$$h(x) = 0$$

$$g(x) \leq 0$$

Example

Consider control problem associated with the linear system

$$x_{k+1} = Ax_k + Bu_k$$

Decision variables: x_k, u_k, $k = 0, 1, \cdots, N$

Cost function:

- x_k is preferred to be close to the origin, the desired steady state.
- Large control action is not desirable.

$$\Downarrow$$

One possible measure of good control is

$$\sum_{i=1}^{N} x_i^T x_i + \sum_{i=0}^{N-1} u_i^T u_i$$

Constraints: decision variables, x_{k+1}, u_k, $k = 0, 1, \cdots, N$, must satisfy the dynamic constraints

$$x_{k+1} = Ax_k + Bu_k$$

$$\Downarrow$$

$$\min_{u_k, x_k} \sum_{i=1}^{N} x_i^T x_i + \sum_{i=0}^{N-1} u_i^T u_i$$

subject to

$$x_{k+1} = Ax_k + Bu_k$$

Terminologies

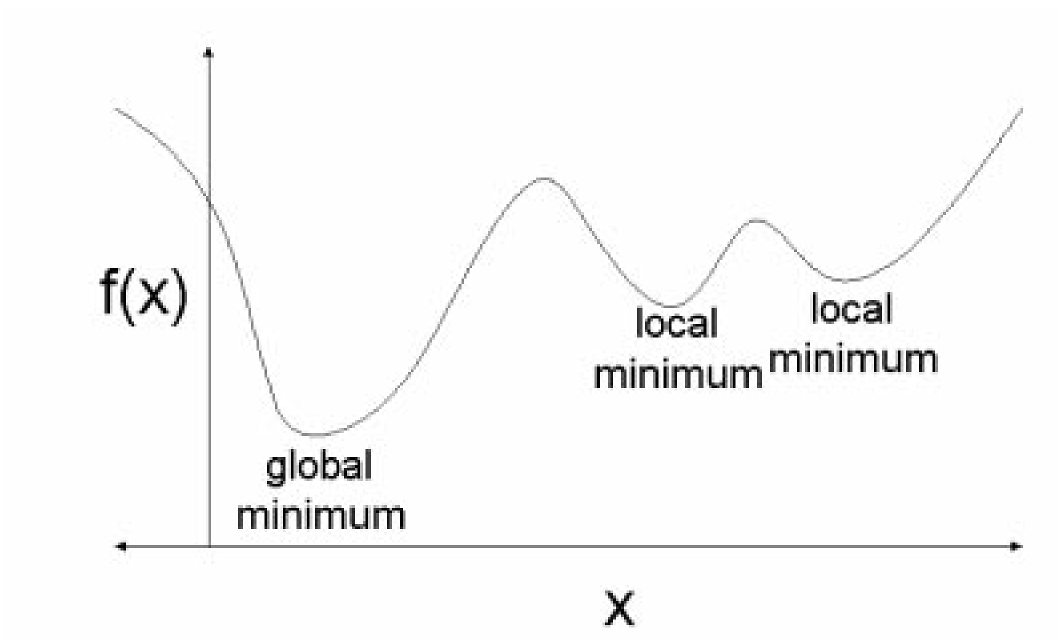
Let

$$\Omega = \{x \in \mathbf{R}^n : h(x) = 0,\ g(x) \le 0\}$$

Feasible point: any $x \in \Omega$

Local minimum: $x^* \in \Omega$ such that $\exists \epsilon > 0$ for which $f(x^*) \le f(x)$ for all $x \in \Omega \cap \{x \in \mathbf{R}^n : \|x - x^*\| < \epsilon\}$.

Global minimum: $x^* \in \Omega$ such that $f(x^*) \le f(x)$ for all $x \in \Omega$.

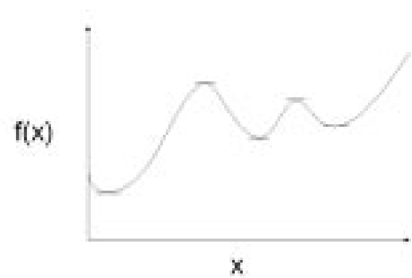


3.2 UNCONSTRAINED OPTIMIZATION PROBLEMS

Necessary Condition of Optimality for Unconstrained Optimization Problems

From calculus, the extrema x^* of a function f from $\mathbf{R}$ to $\mathbf{R}$ must satisfy

$$\frac{df}{dx}(x^*) = 0$$



$$\Downarrow$$

The minima for 1-D unconstrained problem:

$$\min_{x \in \mathbf{R}} f(x)$$

must satisfy

$$\frac{df}{dx}(x^*) = 0$$

that is only necessary.

Necessary Condition of Optimality for Unconstrained Optimization Problems (Continued)

In general, the optima for n-D unconstrained problem:

$$\min_{x \in \mathbf{R}^n} f(x)$$

satisfy the following necessary condition of optimality

$$\nabla f(x^*) = 0$$

(n equations and n unknowns)

Example: Consider

$$\min_{x \in \mathbf{R}^n} \frac{1}{2} x^T H x + g^T x$$

The necessary condition of optimality for this problem is

$$[\nabla f(x^*)]^T = H x^* + g = 0$$

If H is invertible,

$$x^* = -H^{-1} g$$

Steepest Descent Methods

for Unconstrained Nonlinear Programs

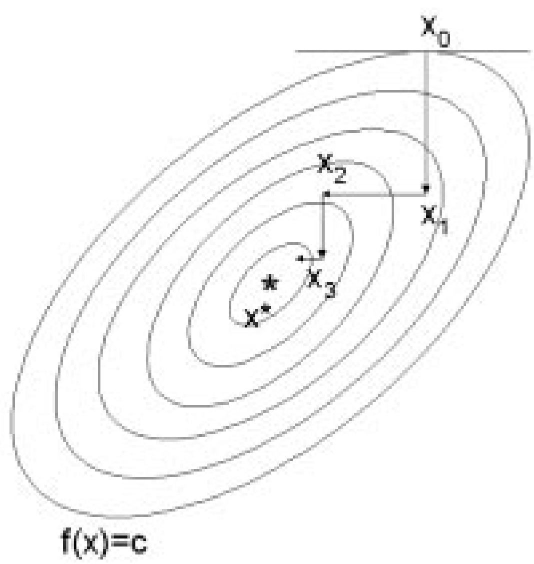
The meaning of gradient $\nabla f(x)$: the steepest ascent direction at the given point.

Main idea: search the minimum in the steepest descnt direction

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

where

$$\alpha_k = \text{argmin}_{\alpha} f(x_k - \alpha \nabla f(x_k))$$



Newton's Method

for Unconstrained Nonlinear Programs

Main idea:

1. Approximate the object function by quadradic function
2. Solve the resulting quadratic problem

Quadratic approximation:

$$f(x) \approx f(x_k) + \nabla f(x_k)(x - x_k) + \frac{1}{2}(x - x_k)^T \nabla^2 f(x_k)(x - x_k)$$

Exact solution of the quadratic program:

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)^T$$