Conduction Heat transfer: Unsteady state
Chapter Objectives

For solving the situations that

– Where temperatures do not change with position.

– In a simple slab geometry where temperature vary also with position.

– Near the surface of a large body (semi-infinite region)
Keywords

- Internal resistance
- External resistance
- Biot number
- Lumped parameter analysis
- 1D and multi-dimensional heat conduction
- Heisler charts
- Semi-infinite region
1 Lumped Parameter Analysis

Figure 1. Several temperatures in the system.

In transient, \( T_{r=r} = T_{r=0.5r} = T_{r=0} \) ?
Lumped Parameter Analysis

Figure 2. A solid with convection over its surface.

\[-mC_p \Delta T = hA(T - T_\infty) \Delta t\]  (1)
Lumped Parameter Analysis

\[-mC_p \Delta T = hA(T - T_\infty) \Delta \Delta \]  \hspace{1cm} (1)

\[\frac{\Delta T}{\Delta t} = \frac{hA}{mc_p}(T - T_\infty)\]

\[\frac{dT}{dt} = -\frac{hA}{mc_p}(T - T_\infty) \]  \hspace{1cm} (2)

\[M : \text{Mass}\]
\[C_p : \text{Specific heat}\]
\[h : \text{Convective heat transfer coefficient}\]
\[A : \text{Surface area}\]
\[T_\infty : \text{Bulk fluid temperature}\]
Lumped Parameter Analysis

\[ T(t = 0) = T_i \]  \hspace{1cm} (3)

\[ \theta = T - T_\infty \]

\[ \theta_i = \theta(t = 0) = T_i - T_\infty \]

\[ \frac{d\theta}{dt} = -\frac{hA}{mc_p} \theta \]
\[ \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = \int_0^t - \frac{hA}{mc_p} \, dt \]

\[ \ln \frac{\theta}{\theta_i} = - \frac{hA}{mc_p} t \]

\[ \frac{\theta}{\theta_i} = e^{- \frac{hA}{mc_p} t} \]

\[ \frac{T - T_\infty}{T_i - T_\infty} = \exp \left( - \frac{hA}{mc_p} t \right) = \exp \left( - \frac{t}{mc_p \frac{1}{hA}} \right) \]
2 Biot Number

![Figure 3. Several temperatures in the system.](image)

So, when can we apply \( T_{r=r} = T_{r=0.5r} = T_{r=0} \)?
Biot Number

Bi (Biot Number): Deciding whether internal resistance can be ignored.

\[ Bi (Biot \ number) = \frac{hL}{k} = \frac{L}{kA} = \frac{1}{hA} = \frac{\text{conductive resistance}}{\text{convective resistance}} \]  \hspace{1cm} (6)

\[ \frac{h(V/A)}{k} < 0.1 \]  \hspace{1cm} (7)
Characteristic Length

Characteristic length \( \equiv \frac{V}{A} \)

Path of least thermal resistance

Characteristic length \( \downarrow \) = Temperature can be changed in short time

\[ \frac{hL}{k} < 0.1 \]  

(8)

Figure 4. Characteristic lengths for heat conduction in various geometries.
Example 1

What is the temperature of the egg after 60min?

Known: Initial temperature of an egg

Find: Temperature of the egg after 60min.

Given data:
- \( T_i \) = 20 °C
- \( T_{air} \) = 38 °C
- \( h \) = 5.2 W/m² · K
- \( \rho \) = 1035 kg/m³
- \( C_p \) = 3350 J/kg · K
- \( k \) = 0.62 W/m · K

Figure 5. Schematic for Example 1.
Assumption:

1. Egg is approximately spherical.

2. Surface heat transfer coefficient provided is an average value.

3. Lumped parameter analysis.

\[ Bi \ (\text{Biot Number}) = \frac{hV}{Ak} = 0.07 < 0.1 \]

**Being Bi <0.1, lumped analysis can be applied!**

Using (Eqn. 5),

\[ \frac{T - T_\infty}{T_i - T_\infty} = \exp \left( -\frac{hA}{mc_p} t \right) \]

\[ \frac{T - 38}{20 - 38} = \exp \left( -\frac{5.2 \text{[W/m}^2 \cdot \text{K]} \times 0.00785 \text{[m}^2]}{1035 \text{[kg/m}^3] \times 60 \times 10^{-6} \text{[m}^3] \times 3350 \text{[J/Kg} \cdot \text{K]} \times 3600 \text{[s]} \right) \]

Then, \( T = 29.1 \, ^\circ \text{C} \)
Assumption:
1. Egg is approximately spherical.
2. Surface heat transfer coefficient provided is an average value.
3. Lumped parameter analysis.

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\]

Then, \( T = 29.1 ^\circ C \)
3 When Internal Resistance Is Not Negligible

The situations, \( T_{r=r} \neq T_{r=0.5r} \neq T_{r=0} \) (i.e. \( \text{Bi} \geq 0.1 \))
When Internal Resistance Is Not Negligible

Figure 6. Schematic of a slab showing the line of symmetry at \( x = 0 \) and the two surfaces at \( x = L \) and at \( x = -L \) maintained at temperature \( T_s \). The material is very large (extends to infinity) in the other two directions.
When Internal Resistance Is Not Negligible

\[
\rho c_p \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = k \left( \frac{\partial^2 T}{\partial x^2} \right) + \dot{Q}^0
\]

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial x^2} \quad (9)
\]

Boundary conditions

\[
\left. \frac{\partial T}{\partial x} \right|_{x = 0, t} = 0 \quad (\text{for symmetry}) \quad (10)
\]

\[
T(L, t > 0) = T_s \quad (11)
\]
When Internal Resistance Is Not Negligible

Initial condition

\[ T(x, t = 0) = T_i \]  \hspace{1cm} (12)

\[ \frac{T - T_s}{T_i - T_s} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n + 1)\pi} \cos\left(\frac{(2n + 1)\pi x}{2L}\right) e^{-\alpha\left(\frac{(2n + 1)\pi}{2L}\right)^2 t} \]  \hspace{1cm} (13)

\[ \alpha \text{ (Thermal diffusivity)} = \frac{k}{\rho C_p} \]
How Temperature Changes with Time

For visualizing Temperature vs. Position and Time, infinite series should be simplified

Figure 7. The terms in the series \((n = 0, 1, \ldots \text{ in Equation 5.13})\) drop off rapidly for values of time. Calculations are for \(F_\circ = 0.0048\) at 30 s and \(F_\circ = 0.096\) at 600 s for a thickness of \(L = 0.03\ m\) and a typical \(\alpha = 1.44 \times 10^{-7}\text{m}^2/\text{s}\) for bio materials.
How Temperature Changes with Time

Comparing different terms at each time ($t = 30s$, $t = 600s$),

Contribution decays

- Gradually at $t = 30s$
- Rapidly at $t = 600s$

\[
\frac{T - T_s}{T_i - T_s} = \frac{4}{\pi} \cos \frac{\pi x}{2L} e^{-\alpha \left(\frac{\pi}{2L}\right)^2 t}
\]  \hspace{1cm} (15)

\[
\ln \frac{T - T_s}{T_i - T_s} = \ln \left(\frac{4}{\pi} \cos \frac{\pi x}{2L}\right) - \alpha \left(\frac{\pi}{2L}\right)^2 t
\]  \hspace{1cm} (16)
Temperature Change with Position and Spatial Average

\[ \frac{T - T_s}{T_i - T_s} = \frac{4}{\pi} \cos \frac{\pi x}{2L} e^{-\alpha \left( \frac{\pi}{2L} \right)^2 t} \]  

(15)

\[ \ln \frac{T - T_s}{T_i - T_s} = \ln \left( \frac{4}{\pi} \cos \frac{\pi x}{2L} \right) - \alpha \left( \frac{\pi}{2L} \right)^2 t \]  

(16)

• We can see that temperature varies as a cosine function

• Therefore, we need to define spatial average temperature
Spatial average temperature

\[ T_{av} = \frac{1}{L} \int_{0}^{L} T \, dx \]  \hspace{1cm} (17)

Applying (5.17) to (5.16) gives

\[ \ln \frac{T_{av} - T_s}{T_i - T_s} = \ln \frac{8}{\pi^2} - \alpha \left( \frac{\pi}{2L} \right)^2 t \]  \hspace{1cm} (18)
\[ \frac{\alpha t}{L^2} = -\frac{4}{\pi^2} \ln \left[ \frac{\pi^2}{8} \left( \frac{T_{av} - T_s}{T_i - T_s} \right) \right] \]  

(19)
Charts Developed from the Solutions: Their Uses and Limitations.

\[ \frac{T - T_s}{T_i - T_s} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n + 1)\pi} \cos \left[ \frac{(2n + 1)\pi}{2} \frac{x}{L} \right] e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 \frac{\alpha t}{L^2}} \] (20)

• It can be seen that temperature is a function of \( x/L \) and \( \alpha t/L^2 \)

• Charts are developed because of the complexity of the calculation of series.
• **Charts are developed with the condition of n=0. In other words, it is a plot of Eqn. 5** And it is also called Heisler chart.

• **There are some assumptions for the development of the charts. These are:**

1. Uniform initial temperature
2. Constant boundary fluid temperature
3. Perfect slab, cylinder or sphere
4. Far from edges
5. No heat generation (Q=0)
6. Constant thermal properties (k, α, c_p are constants)
7. Typically for times long after initial times, given by \( \alpha t/L^2 > 0.2 \)
Figure 8. Unsteady state diffusion in a large slab

For heat transfer,
\[ m = \frac{k}{hL} \]

For mass transfer,
\[ m = \frac{D_{AB}}{h_m L} \quad \text{for } K^* = 1 \]
\[ m = \frac{D_{AB}}{h_m L K^*} \quad \text{for } K^* \neq 1 \]

\[ n = \frac{x}{L} \]

\[ F_0 = \frac{at}{L^2} \quad \text{(heat)} \quad \text{or} \quad \frac{D_{AB} t}{L^2} \quad \text{(mass)} \]
Example 2. Temperatures Reached During Food Sterilization

- Surface temperature of a slab of tuna is suddenly increased
- Find the temperature at the center of the slab after 30 min
• Given data:

1. Thickness of slab = 25 mm
2. Thermal diffusivity of the slab, $\alpha = 2 \times 10^{-7} \text{ m}^2 / \text{s}$
3. Initial temperature = 40°C
4. Surface temperature = 121°C
5. Time of heating = 1800s

• Assumptions

1. Heating from the side is ignored
2. Thermal diffusivity is constant
\[ n = \frac{x}{L} = \frac{0}{0.0125} = 0 \]

\[ m = \frac{k}{hL} = 0 \]

\[ F_0 = \frac{\alpha t}{L^2} = \frac{2 \times 10^{-7} \left[ \text{m}^2 / \text{s} \right] \times 1800 \left[ \text{s} \right]}{(0.0125)^2 \left[ \text{m}^2 \right]} = 2.3 \]

\[ \frac{T - T_\infty}{T_i - T_\infty} = 0.0043 \]

So the temperature \( T = 120.65^\circ \text{C} \) after 30 minutes of heating
Convective Boundary Condition

• We have considered a negligible external fluid resistance to heat transfer.

• But if we consider external fluid resistance in addition to internal fluid resistance,
• At the surface,

\[-k \frac{\partial T}{\partial x} \bigg|_s = h(T_s - T_\infty)\]

The solution is generalized form of Eqn. 5.13 and you can refer to Heisler chart as well.
Numerical Methods as Alternatives to the Charts

• In practice, however, such conditions dealt with above are not that simple

• Limitations of the analytical solutions can be overcome using numerical, computer-based solutions
4 Transient Heat Transfer in a Finite Geometry-Multi-Dimensional Problems

• We should consider the situation two- and three-dimensional effect yields

• A finite geometry is considered as the intersection of two or three infinite geometries

\[
\frac{T_{xyz,t} - T_s}{T_i - T_s} = \left( \frac{T_{x,t} - T_s}{T_i - T_s} \right)_{\text{infinite}} \left( \frac{T_{y,t} - T_s}{T_i - T_s} \right)_{\text{infinite}} \left( \frac{T_{z,t} - T_s}{T_i - T_s} \right)_{\text{infinite}}
\]

(21)
Figure 11. A finite cylinder can be considered as an intersection of an infinite cylinder and a slab

\[
\frac{T_{r,z,t} - T_s}{T_i - T_s} = \left( \frac{T_{r,t} - T_s}{T_i - T_s} \right)_{\text{infinite cylinder}} \left( \frac{T_{z,t} - T_s}{T_i - T_s} \right)_{\text{infinite slab}}
\] (22)
5 Transient Heat Transfer in a Semi-infinite Region

- A semi-infinite region extends to infinity in two directions and a single identifiable surface in the other direction.

- You can see Fig. 5.11 extends to infinity in the y and z directions and has an identifiable surface at x=0.

Figure 12. Schematic of a semi-infinite region showing only one identifiable surface.
• **It can be used practically in heat transfer for a relatively short time and/or in a relatively thick material**

• **The governing equation with no bulk flow and no heat generation is**

\[
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}
\]  

(23)

• **The boundary conditions are**

\[
T(t = 0) = T_i 
\]  

(24)

\[
T(x = 0) = T_s
\]  

(25)

• **The initial condition is**

\[
T(x \to \infty) = T_i
\]  

(26)
• The solution is

\[
\frac{T - T_i}{T_s - T_i} = 1 - \text{erf}\left[\frac{x}{2\sqrt{\alpha t}}\right]
\]  

(27)

The function \( \text{erf}(\eta) \) is called error function and given by

\[
\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} \, d\eta
\]

And here, \( \eta = \frac{x}{2\sqrt{\alpha t}} \)
Figure 13. Comparison of the complementary error function (1-erf(\(\eta\))) with an exponential \(e^{-\eta}\)
• Heat flux at the surface of the semi-infinite region can be calculated with chain rule

\[ q_s'' = -k \frac{dT}{dx} \bigg|_{x=0} \quad = -k \frac{dT}{d\eta} \frac{d\eta}{dx} \bigg|_{x=0} \]

\[ = -k(T_s - T_i) \left( - \frac{2}{\sqrt{\pi}} e^{-\eta^2} \right)_{\eta=0} \frac{1}{2\sqrt{\alpha t}} \]

\[ = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} \]

(28)
The situation we can approximate semi-infinite region

\[
\frac{x}{2\sqrt{\alpha t}} \geq 2 \\
x \geq 4\sqrt{\alpha t}
\]  

(29)

Figure 14. Plot of Eqn. 29, illustrating the minimum thickness of a material for which error function solution can be used.
• Other boundary conditions

1. Convective boundary condition

\[-k \frac{\partial T}{\partial x} \bigg|_{\text{surface}} = h(T_{\text{surface}} - T_{\infty})\]

The solution is

\[
\frac{T - T_i}{T_{\infty} - T_i} = 1 - \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)
\]

\[-e^{\frac{hx}{k} + \frac{h^2\alpha t}{k^2}} \left(1 - \text{erf}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right)\]

(30)
2. Specified surface heat flux boundary condition

\[ q_{\text{surface}}'' = q_s'' \]  \hspace{1cm} (31)

The solution is

\[ T - T_i = \frac{2}{k} q_s'' \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - \frac{q_s'' x}{k} \left( 1 - \text{erf} \left( \frac{x}{2\sqrt{\alpha t}} \right) \right) \]  \hspace{1cm} (32)
Example 3 Analysis of Skin Burns

A thermal burn occurs as a result of an elevation in tissue temperature above a threshold value for a finite period of time.

The intensity of thermal burn is divided into four degrees.
Chapter Summary - Transient Heat Conduction

- **No Internal Resistance, Lumped Parameter**

1. The thermal resistance of the solid can be ignored if a Biot number is less than 0.1.

2. As thermal resistances are ignored, temperature is a function of time only.
• **Internal Resistance is Significant**

1. When internal resistance is significant ($Bi>0.1$), temperature is a function of both position and time

2. For an infinite slab, infinite cylinder and spherical geometry, the solutions are given as Heisler chart. You can find it on pages 327~329.

3. For finite slab and finite cylinder, the solutions are intersection of the infinite slabs and cylinder.

4. Materials with thickness $L \geq 4\sqrt{\alpha t}$ are considered effectively semi-infinite