Chapter 2

Statics, dynamics, and surface tension
Hydrostatics

Fundamental law of hydrostatics for an incompressible fluid
Pressure increases linearly with position in the direction of gravity.

\[ F_{up} = p_2 A = p_1 A + \rho \text{Ahg} = F_{down} \]

\[ p_2 - p_1 = \rho gh \]
What happens if we unplug the bottom of the funnel

Suppose that the exit flow is very slow because the liquid is very viscous or because the exit tube has a very small diameter, and that a drop begins to form slowly at the end of the capillary. Then it is possible that the flow will stop.

Figure 2.1.6 Pressure distribution with and without flow (static pressure profile).
As the drop grows, its weight will overcome the force of surface tension that holds the drop to the tip of the capillary, and the drop will fall off.

Along the three phase contact line, surface tension is the force per unit length acting tangentially to the liquid-gas interface.

There is a vertical component of force

If the drop is static, it balances the weight of the drop

\[ F_\sigma \cos \theta = \sigma \pi D_c \cos \theta \]

\[ \sigma \pi D_c \cos \theta = \rho gV \]
First modeling

Determine the maximum size of a drop that will hang on a capillary

\[ \sigma \pi D_c \cos \theta = \rho g V \]

If the drop volume changes slightly, what happens on the lhs?

As the drop gets heavier, the angle approaches zero, and lhs reaches its maximum. Then the drop will fall when its volume exceeds

\[ V_{\text{max}} = \frac{\sigma \pi D_c}{\rho g} \]

If the drop is nearly a sphere

\[ V = \frac{\pi D_d^3}{6} \]

\[ D_d = \left( \frac{6 \sigma D_c}{\rho g} \right)^{1/3} \]

\[ \frac{D_d}{D_c} = \left( \frac{6}{\text{Bo}} \right)^{1/3} = 1.82 \text{Bo}^{-1/3} \]

\[ \text{Bo} = \frac{D_c^2 \rho g}{\sigma} \]

Bond number: determines the nature of the interaction of gravitational and surface forces
The model captures the essential physics of the process, failing only to yield the exact value of the coefficient. Because it leaves residual liquid attached to the tip when the drop falls from the capillary tip.
Capillary hydrostatics

1. Interfacial tension can give rise to a pressure difference across the interface.
2. The pressure difference can be calculated from the shape and size of the interface.

For any surface, there is a pair of orthogonal curves at any point on the surface.

The interfacial tension is a force per unit length acting across a line element in a surface, the line of action of the force being normal to the line and tangential to the surface.
Young-Laplace equation

Along each of the four arcs that define the differential area, a distributed force acts uniformly.

Sum the force components of these four forces in the direction normal to the surface.

\[ dF_2 = 2(\sigma ds_2)\sin(d\theta_1 / 2) = \frac{\sigma}{R_1} ds_1 ds_2 \]

\[ ds_1 = R_1 d\theta_1 \quad \sin\left(\frac{1}{2} d\theta_1\right) = \frac{1}{2} d\theta_1 \]

for small angle

Static force balance

\[ p_o - p_i + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) ds_1 ds_2 = 0 \]

\[ p_i - p_o = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \]
Pressure inside a gas bubble

For a spherical gas bubble

\[ R_1 = R_2 = R \quad \Delta p = \frac{2\sigma}{R} \]

Typical surface tension of a liquid is 0.05 N/m

For a bubble of radius 1 mm

\[ \Delta p = \frac{2(0.05)}{0.001} = 100 \text{ N/m}^2 \]

For a bubble of radius 1 μm

\[ \Delta p = \frac{2(0.05)}{1 \times 10^{-6}} = 1 \times 10^5 \text{ N/m}^2 \approx 1 \text{ atm} \]

Exercises

Liquid rise in a capillary

Back-pressure in a growing drop
Pressure in response to external forces

Differential force balance in z direction

\[ dp_z = -\rho g \, dz \]

\[ p_2 - p_1 = \rho gh \]

Differential force balance in r direction

\[ dp_r \, rd \theta dh = dm \times a = dm \, r \omega^2 = \rho \, rd \theta dr dh \, r \omega^2 \]

\[ dp = \frac{\partial p}{\partial z} \, dz + \frac{\partial p}{\partial r} \, dr = dp_z + dp_r \]

Pressure field within a rotating body of liquid

\[ p = -\rho gz + \frac{\rho r^2 \omega^2}{2} \]
Surface of a rotating liquid

Set atmospheric pressure to zero

\[ p = -\rho g z + \frac{\rho r^2 \omega^2}{2} \]

Pressure in an accelerating liquid: shape of the free surface

Liquid container moves in the +x direction at a velocity that increases linearly with time

\[ p_x \, dA - p_{x+dx} \, dA = \rho (dA)(dx) a_o \]

\[ \frac{\partial p}{\partial x} = -\rho a_o \quad p(x) = -\rho a_o x + A(y) \]

\[ \frac{\partial p}{\partial y} = -\rho g \quad p(y) = -\rho gy + B(x) \]

\[ p(x, y) = -\rho a_o x - \rho gy + D \]

Atmospheric pressure at surface \( Y(x) \)

\[ Y(x) = -\frac{(p_o - D) + \rho a_o x}{\rho g} \]