Some Reviews and Exercises on Sections 9.1. ~ 9.6.

Section 9.2. Line integral independent of path

Line integral independent of path:

End points

$$f(x,y); \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Pdx + Qdy \Rightarrow \int_C (Pdx + Qdy) = f(B) - f(A)$$

exact differential

Test for path independence in plane: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Ex.) $\displaystyle \int_C (y^2 - 6xy + 6)dx + (2xy - 3x^2)dy; \ A(-1,0) \to B(3,4)$

$$\int_{(-1,0)}^{(3,4)} d(xy^2 - 3x^2y + 6x) = -36$$

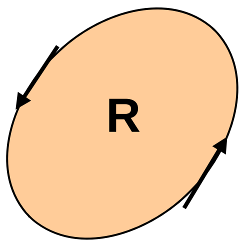
$\underline{\nabla} \times \underline{F} = \underline{0}$

Test for path independence in space: $\displaystyle \int_C Pdx + Qdy + Rdz; \ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

Section 9.4: Green’s Theorem

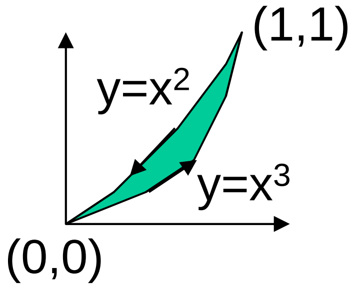
$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

$$\iint_R (\underline{\nabla} \times \underline{F}) \cdot \underline{k} \, dxdy = \oint_C \underline{F} \cdot d\underline{r} \quad (\underline{F} = F_1 \underline{i} + F_2 \underline{j})$$



Positive direction

Ex.) $\oint_C (x^2 - y^2)dx + (2y - x)dy;$ C bounded with $y = x^2$ & $y = x^3$

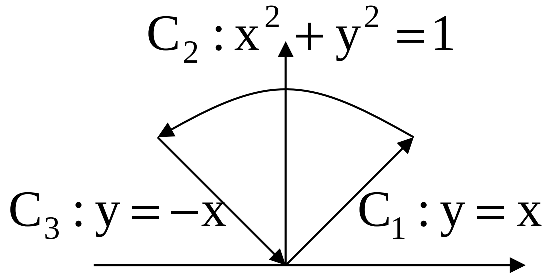


$$= \iint_R (-1 + 2y)dA = \int_0^1 \int_{x^3}^{x^2} (-1 + 2y)dydx = -11/420$$

Ex.) $\oint_C (x^5 + 3y)dx + (2x - e^{y^3})dy;\ C: (x-1)^2 + (y-5)^2 = 4$

$$= \iint_R (2-3)dA = -\iint_R dA = -4\pi$$

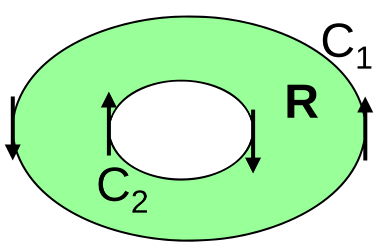
Ex.) $F = (-16y + \sin x^2)\underline{i} + (4e^y + 3x^2)\underline{j};\ W = \oint_C \underline{F} \cdot d\underline{r}$

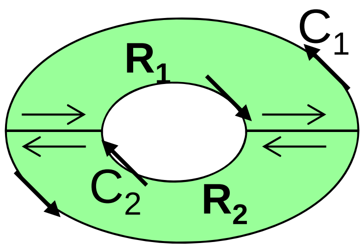


$$W = \oint_C (-16y + \sin x^2)dx + (4e^y + 3x^2)dy = \iint_R (6x + 16)dA$$

$$= \int_{\pi/4}^{3\pi/4} \int_0^1 (6r\cos\theta + 16)rdrd\theta = 4\pi$$

Region with Holes



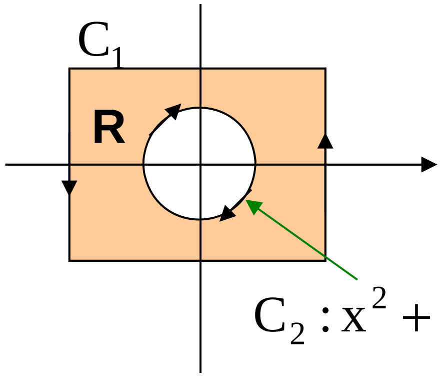


$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_{R_1} (\quad) dA + \iint_{R_2} (\quad) dA$$

$$= \oint_{C_1} F_1 dx + F_2 dy + \oint_{C_2} F_1 dx + F_2 dy = \oint_C F_1 dx + F_2 dy$$

$$\left(C = C_1 \cup C_2 \right)$$

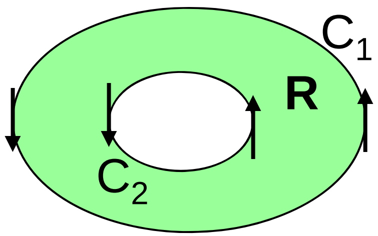
Ex.) $\oint_C \left(-\frac{y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy;\ \ C = C_1 \cup C_2$



$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2} : \text{continuous on region R}$$

(at origin?)

$$\oint_C (\quad)dx + (\quad)dy = \iint_R (\quad)dA = 0$$



F_1 and F_2 have continuous first partial derivatives such that

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\oint_{C_1}(F_1 dx + F_2 dy) + \oint_{-C_2}(F_1 dx + F_2 dy) = 0 \Rightarrow \oint_{C_1}(F_1 dx + F_2 dy) = \oint_{C_2}(F_1 dx + F_2 dy)$$

Section 9.6. Surface Integrals

Integrals of vector fields: $\iint_S \underline{F} \cdot \underline{n}\, dA = \iint_R \underline{F}(\underline{r}(u,v)) \cdot \underline{N}(u,v)\, du dv$ "Flux integral"

Ex.) $F(x,y,z) = z\underline{j} + z\underline{k},\ \ g(x,y,z) = 3x + 2y + z - 6 = 0,\ \ \underline{n} = \frac{\underline{\nabla g}}{|\underline{\nabla g}|} = \frac{3}{\sqrt{14}}\underline{i} + \frac{2}{\sqrt{14}}\underline{j} + \frac{1}{\sqrt{14}}\underline{k},$

$$\text{flux} = \iint_S (F \cdot n) dA = \frac{1}{\sqrt{14}} \iint_S 3z dA = \frac{1}{\sqrt{14}} \int_0^2 \int_0^{3-3x/2} 3(6 - 3x - 2y)\sqrt{14}\, dy dx = 18$$

Method of evaluation for surface Integrals   z=f(x,y)

$$\iint_S G(\underline{r})dA = \iint_{R^*} G(x,y,f(x,y)) \left|\sqrt{1+(\partial f/\partial x)^2+(\partial f/\partial y)^2}\right| dxdy$$

$$A(S) = \iint_{R^*} \left|\sqrt{1+(\partial f/\partial x)^2+(\partial f/\partial y)^2}\right| dxdy$$

(R^*: projection of S into xy plane)

Projection of S into other planes

y=g(x,z): *Eqn. of a surface S that projects onto a region R^* of the xz-plane*

$$\iint_S G(\underline{r})dA = \iint_{R^*} G(x,g(x,z),z) \left|\sqrt{1+(\partial g/\partial x)^2+(\partial g/\partial z)^2}\right| dxdz$$

x=h(y,z)

$$\iint_S G(\underline{r})dA = \iint_{R^*} G(h(y,z),y,z) \left|\sqrt{1+(\partial h/\partial y)^2+(\partial h/\partial z)^2}\right| dydz$$

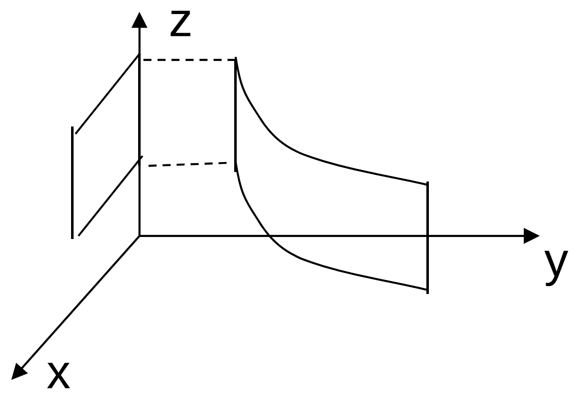
Ex.) Find the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ that is above the xy-plane and within the cylinder $x^2 + y^2 = b^2$, $0<b<a$.

$$z = f(x,y) = \sqrt{a^2 - x^2 - y^2},\ f_x = -\frac{x}{\sqrt{a^2 - x^2 - y^2}},\ f_y = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

$$A(S) = \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy = a\int_0^{2\pi}\int_0^b (a^2 - r^2)^{-1/2} r\, drd\theta = 2\pi a(a - \sqrt{a^2 - b^2})$$

Ex.) $\iint_S xz^2 dA,\ S: y = 2x^2 + 1\,(0 \le x \le 2, 4 \le z \le 8)$

$$\Rightarrow \int_0^2 \int_4^8 xz^2 \sqrt{1+16x^2}\, dzdx = \frac{28}{9}\left(65^{3/2} - 1\right)$$



9.7. Triple Integrals: Gauss's Divergence Theorem

- Relationship btw. *triple integrals* and *surface integrals*

- Triple integrals of f(x,y,z) over the region T

$$\iiint_T f(x,y,z)\,dxdydz \quad \text{or} \quad \iiint_T f(x,y,z)\,dV$$

Gauss's Divergence Theorem

- Well used in fluid mechanics, heat transfer, …

- Divergence: $\underline{\nabla}\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Theorem 1: Gauss's Divergence Theorem

T: closed bounded region in space, S: orientable boundary surface

$$\iiint_T \underline{\nabla}\cdot\underline{F}\,dV = \iint_S \underline{F}\cdot\underline{n}\,dA$$ (n: outer unit normal vector of S)

$$n = \cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA$$

$$= \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

Ex. 1) $I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 dxdy\right); \;\; x^2 + y^2 = a^2 \;(0 \le z \le b)$

$$\underline{\nabla} \cdot \underline{F} = 3x^2 + x^2 + x^2 = 5x^2$$

$$\Rightarrow \iiint_T 5x^2 dxdydz = \int_0^b \int_0^{2\pi} \int_0^a 5r^2 \cos^2\theta r \, dr d\theta dz = \frac{5}{4}\pi b a^4$$

$$\iiint_T \frac{\partial F_1}{\partial x} dxdydz = \iint_S F_1 \cos\alpha \, dA, \cdots, \;\; \iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 \cos\gamma \, dA$$

See Fig. 231

$g(x,y) \le z \le h(x,y)$ *((x,y) varying in the orthogonal projection R of T in the xy-plane)*

$$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_R \left[\int_{g(x,y)}^{h(x,y)} \frac{\partial F_3}{\partial z} dz\right] dxdy = \iint_R \left[F_3(x,y,\underset{\substack{\uparrow \\ \cos\gamma > 0}}{h(x,y)}) - F_3(x,y,\underset{\substack{\uparrow \\ \cos\gamma < 0}}{g(x,y)})\right] dxdy$$

$$= \iint_S F_3 \cos\gamma dA = \iint_S F_3 dxdy$$

Ex. 2) $\iint_S (7x\underline{i} - z\underline{k}) \cdot \underline{n} \, dA; \;\; S: x^2 + y^2 + z^2 = 4$

$$\underline{\nabla} \cdot \underline{F} = 6, \;\; \iiint_T 6dV = 6\left(\frac{4}{3}\pi 2^3\right) = 64\pi$$