Chap. 11. PARTIAL DIFFERENTIAL EQUATIONS

- An equation involving partial derivatives of an unknown function of two more independent variables → PDE

Classification of PDES

- **Linear and nonlinear PDEs**

Linear PDE: There is no product of the dependent variable and/or product of its derivatives within the equation

Nonlinear PDE: The equation contains a product of the dependent variable and/or a product of the derivatives

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1 \text{ (2nd} - \text{order, linear)}, \quad \frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y \text{ (3rd} - \text{order, linear)}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x \text{ (nonlinear)}, \quad \frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = x \text{ (nonlinear)}$$

- **Classification based on characteristics (paths of propagation of physical disturbances)**

(I) First-order PDE: Almost all first-order PDEs have real characteristics, and therefore behave much like hyperbolic equations of second order.

(II) Second-order PDE: A second-order PDE in two dependent variables, x and y, may be expressed in a general form as

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} + D\frac{\partial \phi}{\partial x} + E\frac{\partial \phi}{\partial y} + F\phi + G = 0$$

- **The equation is classified according to the expression (B^2-4AC) as follows:**

$(B^2-4AC) < 0 \rightarrow$ Elliptic equation
$= 0 \rightarrow$ Parabolic equation
$> 0 \rightarrow$ Hyperbolic equation

(a) Elliptic equations

No real characteristic lines exist
A disturbance propagates in all directions
Domain of solution is a closed region
Boundary conditions must be specified on the boundaries of the domain

(b) Parabolic equations

Only one characteristic line exists
A disturbance propagates along the characteristic line
Domain of solution is an open region
An initial condition and two boundary conditions are required

(c) Hyperbolic equations

Two characteristic lines exist
A disturbance propagates along the characteristic lines
Domain of solution is an open region
Two initial conditions along with two boundary conditions are required

- **Boundary conditions**

(a) Dirichlet B.C. (=Essential B.C.): The value of the dependent variable along the boundary is specified
(b) Neumann B.C (=Natural B.C.): The normal gradient of the dependent variable along with the boundary is specified
(c) Mixed B.C. (Robbin B.C.): A combination of the Dirichlet and the Neumann type B.C.'s is specified

•11.1. Basic Concepts

- Linear & nonlinear

- Homogeneous & nonhomogeneous

Ex.1) Important linear 2nd-order PDEs

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *1D wave Eqn.*

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *1D heat Eqn.*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ *2D Laplace Eqn.*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ *2D Poisson Eqn.*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ *2D wave Eqn.*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ *3D Laplace Eqn.*

Theorem 1: **Superposition or linearity principle**

u_1, u_2: solutions of a linear homogeneous PDE in R, then

$u = c_1 u_1 + c_1 u_2$: also solution of that equation in R

Ex. 1) A solution u(x,y) of PDE $u_{xx}-u=0$

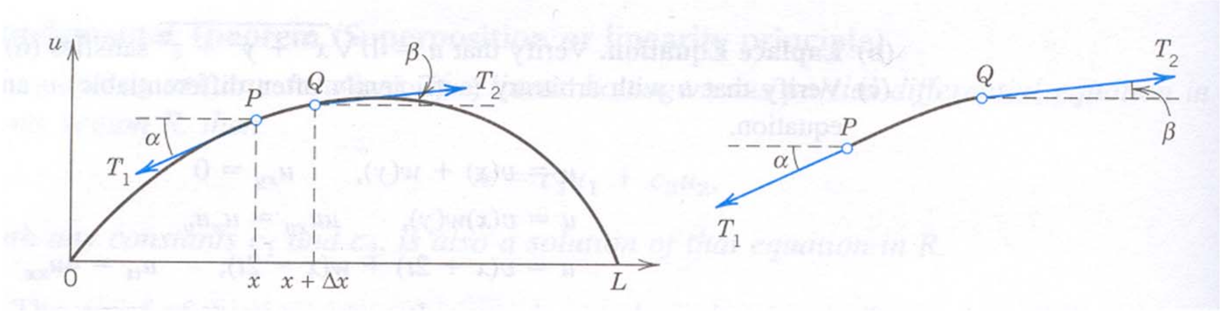
$u(x,y) = A(y)e^x + B(y)e^{-x}$

Ex. 2) PDE $u_{xy} = -u_x$

$u_x = p$ → $p_y=-p$: $p=c(x)e^{-y}$ → $u(x,y) = f(x)e^{-y} + g(y)$ where $f(x) = \int c(x)dx$

11.2. Modeling: Vibrating String, Wave Equation

- Equation governing small transverse vibration of an elastic string
  Find the deflection u(x,t):



Assumptions: - Constant mass/unit length, perfect elastic, no resistance to bending
- Negligible gravitational force
- Small transverse motion in vertical plane → vertical movement

Derivation of the PDE from forces

In horizontal direction: $T_1\cos\alpha = T_2\cos\beta = T = \text{const}$

In vertical direction: $T_2\sin\beta - T_1\sin\alpha = \rho\Delta x\,\frac{\partial^2 u}{\partial t^2}$

$$\frac{T_2\sin\beta}{T_2\cos\beta}-\frac{T_1\sin\alpha}{T_1\cos\alpha}=\underset{\left.\left(\frac{\partial u}{\partial x}\right)\right|_{x+\Delta x}}{\underset{\uparrow}{\tan\beta}}-\underset{\left.\left(\frac{\partial u}{\partial x}\right)\right|_{x}}{\underset{\uparrow}{\tan\alpha}}=\frac{\rho\Delta x}{T}\,\frac{\partial^2 u}{\partial t^2}$$

1D Wave Equation:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}\quad\left(c^2=\frac{T}{\rho}\right)$$

2nd-order Hyperbolic PDE

11.3. Separation of Variables: Use of Fourier Series

- 1D Wave equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \quad \left(c^2 = \dfrac{T}{\rho} \right)$

2 B.C.'s: u(0,t) = u(L,t) = 0 for all t

2 I.C.'s : u(x,0) = f(x), $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x)$

Initial deflection *Initial velocity*

Solving Steps: - Method of separation variables leading to two ODEs.
- Solutions of two eqns. satisfying B.C's
- Final solution of wave eqn. satisfying I.C's, using Fourier series

First Step: Two ODEs using method of separation variables

- u(x,t) = F(x)G(t) ⇒

(linear system)

(derivative w.r.t t) *(derivative w.r.t x)*

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k = \text{const} \Rightarrow \quad \begin{matrix} F'' - kF = 0 \\ \ddot{G} - c^2 kG = 0 \end{matrix}$$

Second Step: Satisfying the B.C.'s

- u(0,t) = F(0)G(t) = 0 Case 1) G = 0 → u = 0 (∴G≠0) Case 2) k=0 → F=0 (∴k≠0)

u(L,t) = F(L)G(t) = 0 Case 3) k=μ^2 → F=0 ⇒ ∴ k = -p^2 *(negative)*

Solving F(x):

$$F''+p^2F=0 \;\Rightarrow\; F(x)=\cancel{A}\cos px+B\sin px \;\leftarrow \text{apply B.C's}: F(0)=F(L)=0$$

$$\Rightarrow F_n(x)=\sin\frac{n\pi}{L}x \quad \text{for } B=1 \;\; (n=1,2,\cdots) \quad (p=n\pi/L)$$

Solving G(t): $\ddot{G}+\lambda_n^2G=0 \quad \left(\lambda=\frac{cn\pi}{L}\right) \quad \Rightarrow G_n(t)=B_n\cos\lambda_n t+B_n^*\sin\lambda_n t$

$$u_n(x,t)=\left(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t\right)\left(\sin\frac{n\pi}{L}x\right) \quad (n=1,2,\cdots)$$

↑

(Eigenfunctions or characteristic functions) *(λ_n: eigenvalues or characteristic values)*

U_n: harmonic motion with frequency $\lambda_n/2\pi=cn/2L$ (n^th normal mode)
n^th normal mode has n-1 nodes

Tuning controlled by tension T (or $c^2=T/\rho$)

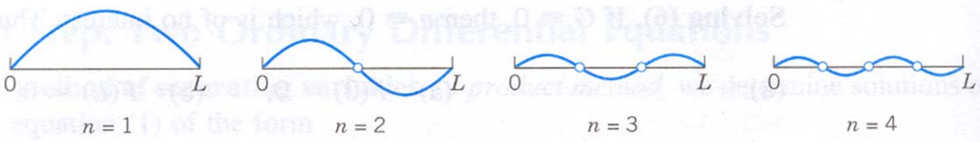


Fig. 259. Normal modes of the vibrating string

Standing wave solutions

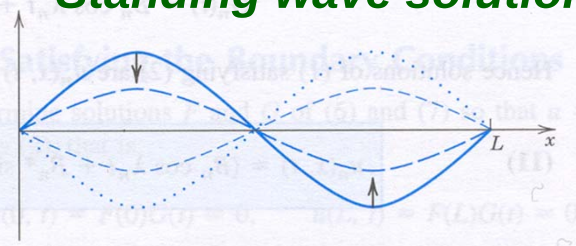


Fig. 260. Second normal mode for various values of *t*

Third Step: Solution to the Entire Problem. Fourier Series

- Sum of many solutions u_n satisfying I.C.'s:

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t\right)\left(\sin\frac{n\pi}{L}x\right)$$

- **Satisfying I.C.1:** initial displacement (u(x,0) = f(x))

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{L}x=f(x)\qquad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

(Fourier sine series)

- **Satisfying I.C.2:** initial velocity $\left(\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)\right)$

(Fourier sine series)

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\sum_{n=1}^{\infty}B_n^*\lambda_n\sin\frac{n\pi}{L}x=g(x)\qquad B_n^*\lambda_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

- **Solution (I):** for the simple case of g(x) = 0

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin\frac{n\pi}{L}x\qquad\left(\lambda_n=\frac{cn\pi}{L}\right)$$

(f: odd periodic extension of f with period 2L)*

$$=\frac{1}{2}\sum_{n=1}^{\infty}B_n\sin\left\{\frac{n\pi}{L}(x-ct)\right\}+\frac{1}{2}\sum_{n=1}^{\infty}B_n\sin\left\{\frac{n\pi}{L}(x+ct)\right\}=\frac{1}{2}\left[f^*(x-ct)+f^*(x+ct)\right]$$

Odd periodic extension of f(x)

Fig. 261. Odd periodic extension of $f(x)$

Physical Interpretation of the Solution

f*(x - ct): a wave traveling to the right as t increases

constant along each line x - ct

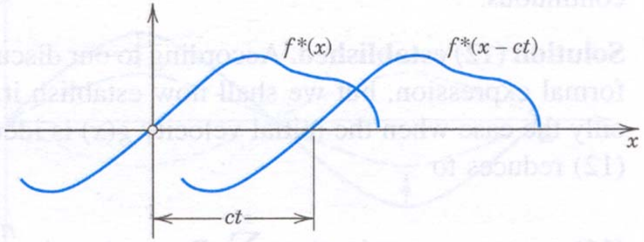


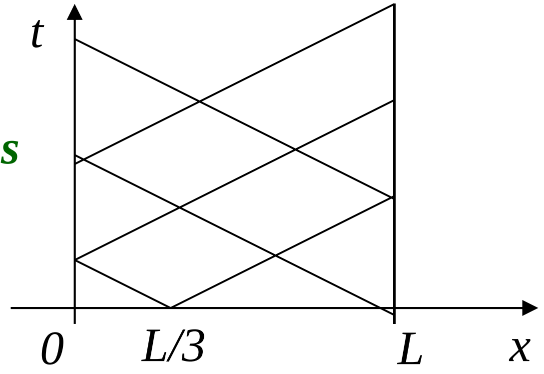
Fig. 262. Interpretation of (17)

f*(x + ct): a wave traveling to the left as t increases

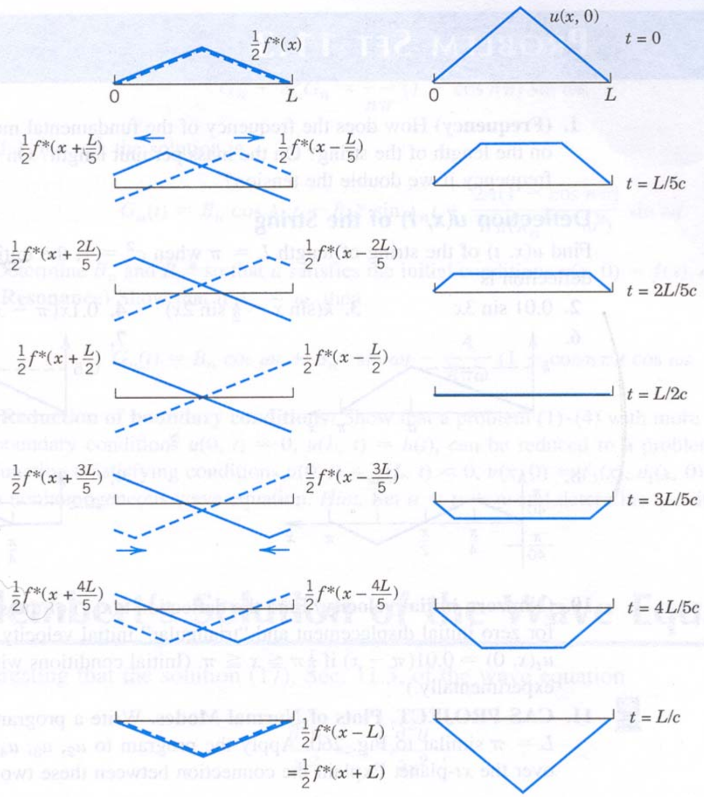
constant along each line x + ct

c: wave velocity

→ u(x,t): superposition of above two waves



characteristic lines



Ex. 1) Vibrating string

if the initial deflection is triangular.

See Ex. 3 in Sec. 10.4

Solution (II): for the case of f(x)=0

$$u(x,t)=\sum_{n=1}^{\infty} B_n^* \sin\lambda_n t \sin\frac{n\pi}{L}x \qquad \left(\lambda_n=\frac{cn\pi}{L}\right)$$

$$=\frac{1}{2}\sum_{n=1}^{\infty} B_n^* \cos\left\{\frac{n\pi}{L}(x-ct)\right\}-\frac{1}{2}\sum_{n=1}^{\infty} B_n^* \cos\left\{\frac{n\pi}{L}(x+ct)\right\}=\frac{1}{2c}\left[G(x+ct)-G(x-ct)\right]$$

$$\left(g(x)=\sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n\pi}{L}x,\ \ G(x)=-\sum_{n=1}^{\infty} B_n^* c\cos\frac{n\pi}{L}x \Rightarrow G'(x)=g(x)\right)$$

Solution (III): for the general case of f(x)≠0 and g(x)≠0

$$u(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\left[G(x+ct)-G(x-ct)\right]$$

$$=P(x+ct)+Q(x-ct)$$

Exercise: Find the solution of the wave equation with following B.C.'s & I.C.'s

$u_{tt} = c^2u_{xx}$

B.C.'s: $u_x(0,t) = u_x(\pi,t) = 0$ for all t

I.C.'s: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

(use Fourier cosine series)

11.4. D'Alembert's Solution of the Wave Equation

- Other method to obtain the solution of the wave eqn. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \left(c^2 = \frac{T}{\rho} \right)$

u(x,t) → u(v,z) using v = x + ct, z = x – ct

$$u_x = \frac{\partial u}{\partial x} = u_v v_x + u_z z_x, \; u_{xx} = \frac{\partial}{\partial x}(u_x) = u_{vv} + 2u_{vz} + u_{zz}, \; u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz})$$

➡ $$c^2(u_{vv} - 2u_{vz} + u_{zz}) = c^2(u_{vv} + 2u_{vz} + u_{zz}) \quad \Rightarrow u_{vz} = \frac{\partial^2 u}{\partial z \partial v} = 0$$

$$\frac{\partial u}{\partial v} = h(v) \to u = \int h(v)dv + \psi(z) = \phi(v) + \psi(z) \;\Rightarrow\; u(x,t) = \phi(x+ct) + \psi(x-ct)$$

(D'Alembert's solution)

D'Alembert Solution Satisfying the Initial Conditions

$$u(x,0) = \phi(x) + \psi(x) = f(x) \qquad \left(k(x_0) = \phi(x_0) - \psi(x_0) \right)$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x) \;\to\; \phi(x) - \psi(x) = k(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + \frac{1}{2}k(x_0)$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - \frac{1}{2}k(x_0)$$

➡ $$u(x,t) = \boxed{\frac{1}{2}\left[f(x+ct) + f(x-ct)\right]} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

⮤ if g(s)=0