10.8. Fourier Integrals

- Application of Fourier series to *nonperiodic function*
  Use Fourier series of a function f_L with period L ($L \rightarrow \infty$)

Ex. 1) Square wave

$$f_L(x) = \begin{cases} 0 \text{ if } -L < x < -1 \\ 1 \text{ if } -1 < x < 1 \\ 0 \text{ if } 1 < x < L \end{cases} \quad (2L > 2)$$

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 \text{ if } -1 < x < 1 \\ 0 \text{ other regions} \end{cases}$$

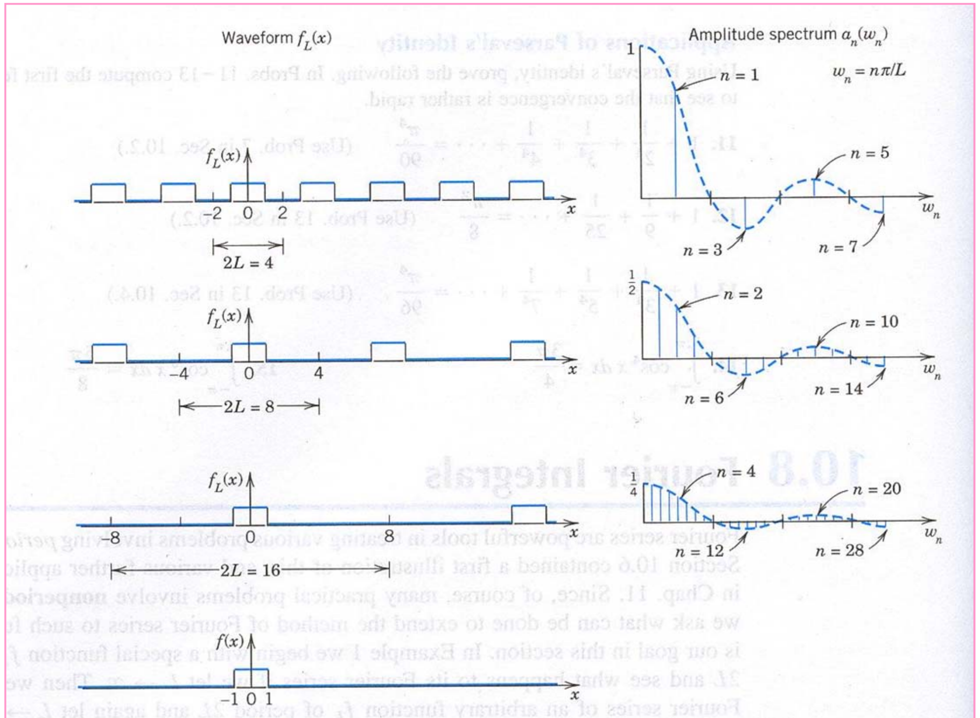

Fig. 254. Waveforms and amplitude spectra in Example 1

$$\left(a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L},\ a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L} \right) \longleftarrow \textit{Amplitude spectrum}$$

From Fourier Series to the Fourier Integral

Fourier series of f(x) (period 2L): $f_L(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos w_n x + b_n \sin w_n x\right), \quad w_n = \frac{n\pi}{L}$

L → ∞, f(x) ?

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$\Delta w = w_{n+1} - w_n = \frac{\pi}{L} \quad \left(\frac{1}{L} = \frac{\Delta w}{\pi}\right)$$

$$\Rightarrow f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\begin{array}{l}(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v dv \\ + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v dv\end{array}\right]$$

If $f(x) = \lim_{L\to\infty} f_L(x)$ is absolutely integrable, $\int_{-\infty}^{\infty} |f(x)| dx$ exists

L → ∞, then $f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv dv\right]dw$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv$$

$$f(x) = \int_0^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right]dw$$

Fourier Integral

Theorem 1: Fourier Integral

- f(x): piecewise continuous
  right-hand / left-hand derivatives exist
  integral $\int_{-\infty}^{\infty} |f(x)| dx$ exists

⇒ f(x) can be represented by Fourier integral.

Applications of the Fourier Integral

- Solving differential equations (*see 11.6*) & integration, …

Ex. 2) Single pulse, sine integral

$$f(x) = 1 \text{ if } |x| < 1, \; 0 \text{ if } |x| > 1$$

$$A(w) = \frac{1}{\pi}\int_{-1}^{1} \cos wv\,dv = \frac{2\sin w}{\pi w}, \quad B(w) = \frac{1}{\pi}\int_{-1}^{1} \sin wv\,dv = 0$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos wx \sin w}{w} dw$$

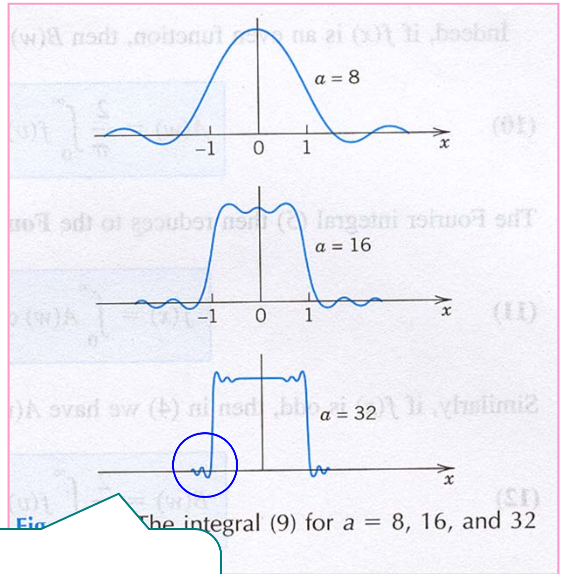

Fig. ... The integral (9) for a = 8, 16, and 32

Fluctuation caused by the Si func

$$\int_{0}^{\infty} \frac{\cos wx \sin w}{w} dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Dirichlet's discontinuous factor

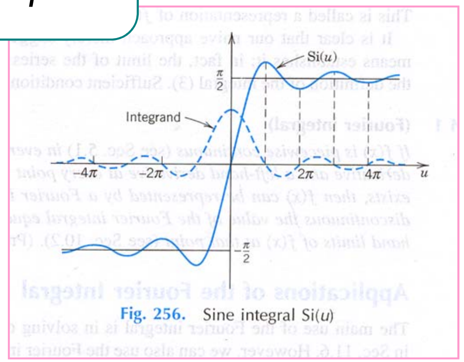

Fig. 256. Sine integral Si(u)

$$\int_{0}^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2} \text{ at } x = 0$$

Sine integral: $\mathrm{Si}(u) = \int_{0}^{u} \frac{\sin w}{w} dw$

Fourier Cosine and Sine Integrals

For even function f(x): B(w)=0, $A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos wv dv$

For odd function f(x): A(w)=0, $B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin wv dv$

Fourier cosine integral:

$$f(x)=\int_0^\infty A(w)\cos wx\, dw$$

Fourier sine integral:

$$f(x)=\int_0^\infty B(w)\sin wx\, dw$$

Evaluation of Integrals

- Fourier integrals for evaluating integrals

Ex. 3) Laplace integrals

$$f(x)=e^{-kx} \quad (x, k>0)$$

(a) Fourier cosine integral: $A(w)=\frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv dv=\frac{2k/\pi}{k^2+w^2}$

$$f(x)=e^{-kx}=\frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}dw \Rightarrow \int_0^\infty \frac{\cos wx}{k^2+w^2}dw=\frac{\pi}{2k}e^{-kx}$$

(b) Fourier sine integral: $B(w)=\frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv dv=\frac{2w/\pi}{k^2+w^2}$

$$f(x)=e^{-kx}=\frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}dw \Rightarrow \int_0^\infty \frac{w\sin wx}{k^2+w^2}dw=\frac{\pi}{2}e^{-kx}$$

10.9. Fourier Cosine and Sine Transforms

- Integral transforms: useful tools in solving ODEs, PDEs, integral equations, and special functions …

Laplace transforms
Fourier transforms ← from Fourier integral expressions

- Fourier cosine transforms, Fourier sine transforms (for real…)
  Fourier transforms (for complex…)

Fourier Cosine Transforms

Fourier cosine integral for even function f(x): $f(x) = \int_0^\infty A(w)\cos wx\, dw$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv dv, \quad A(w) = \sqrt{\frac{2}{\pi}} F_C(w)$$

$$F_C(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx : f(x) \to F_C(w)$$

Fourier cosine transform of f(x)

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(w)\cos wx dw : \; F_C(w) \to f(x)$$

Inverse Fourier cosine transform of $F_C(x)$

Fourier Sine Transforms

Fourier sine integral for even function f(x): $f(x) = \int_0^\infty B(w)\sin wx\,dw$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv dv, \quad B(w) = \sqrt{\frac{2}{\pi}} F_S(w)$$

$$F_S(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx dx : f(x) \to F_S(w)$$

Fourier sine transform of f(x)

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S(w)\sin wx dw : \; F_S(w) \to f(x)$$

Inverse Fourier sine transform of $F_C(x)$

Ex. 1) Fourier cosine and sine transforms

$$f(x) = k, \; 0 < x < a; 0, \; x > a$$

See Table I & II (in 10.11)

$$F_C(w) = \sqrt{\frac{2}{\pi}}\int_0^a k\cos wx dx = \sqrt{\frac{2}{\pi}} k\left(\frac{\sin aw}{w}\right), \quad F_S(w) = \sqrt{\frac{2}{\pi}}\int_0^a k\sin wx dx = \sqrt{\frac{2}{\pi}} k\left(\frac{1-\cos aw}{w}\right)$$

Ex. 2) Fourier cosine transform of the exponential function: $f(x) = e^{-x}$

$$F_C(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos wx dx = \frac{\sqrt{2/\pi}}{1+w^2}$$

Linearity, Transforms of Derivatives

$$\Im_C(af+bg)=\sqrt{\frac{2}{\pi}}\int_0^\infty (af+bg)\cos wx\,dx$$

$$\Im_C(f)=F_C,\ \ \Im_S(f)=F_S$$

$$\Im_C(af+bg)=a\Im_C(f)+b\Im_C(g);\ \ \Im_S(af+bg)=a\Im_S(f)+b\Im_S(g)$$

Theorem 1: Cosine and sine transforms of derivatives

f(x): continuous & absolutely integrable, f(x) → 0 as x → ∞

f'(x) piecewise continuous

$$\Im_C(f'(x))=w\Im_S(f(x))-\sqrt{\frac{2}{\pi}}f(0);\ \ \ \Im_S(f'(x))=-w\Im_C(f(x))$$

$$\Im_C(f''(x))=-w^2\Im_C(f(x))-\sqrt{\frac{2}{\pi}}f'(0);\ \ \ \Im_S(f''(x))=-w^2\Im_S(f(x))+\sqrt{\frac{2}{\pi}}wf(0)$$

$$\Im_C(f'(x))=\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx=\sqrt{\frac{2}{\pi}}\left[f(x)\cos wx\Big|_0^\infty+w\int_0^\infty f(x)\sin wx\,dx\right]=\cdots$$

$$\Im_C(f''(x))=w\Im_S(f'(x))-\sqrt{\frac{2}{\pi}}f'(0)\leftarrow\Im_S(f'(x))=-w\Im_C(f(x))$$ ***(See section 11.6)***

Ex. 3) Fourier cosine transforms of exp. Function: $f(x)=e^{-ax}$ (a > 0)

$$\Im_C(f'')=a^2\Im_C(f)=-w^2\Im_C(f)+a\sqrt{\frac{2}{\pi}}\Rightarrow\Im_C(f)=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right)$$

10.10. Fourier Transform

- from *Fourier integral* in complex form

Complex Form of the Fourier Integral

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)(\cos wv \cos wx + \sin wv \sin wx)dvdw$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right]dw$$

even function of w !

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right]dw$$

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)dv\right]dw = 0\right)$$

Use Euler's formula: $e^{ix} = \cos x + i\sin x$

$$\Rightarrow f(v)[\cos(wx - wv) + i\sin(wx - wv)] = f(v)e^{i(wx - wv)}$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)e^{i(wx-wv)}dv\right]dw$$

Fourier Transform and Its Inverse

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-iwv}dv\right]e^{iwx}dw$$

Fourier transform of f(x): $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$

Inverse Fourier transform of F(w): $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(w)e^{iwx}dw$

Ex. 1) Fourier transform of f(x)=k (0<x<a) and f(x)=0 otherwise

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{0}^{a}ke^{-iwx}dx=\frac{k(1-e^{-aw})}{iw\sqrt{2\pi}}$$

Ex. 2) Fourier transform of $f(x)=e^{-ax^2}$ (a>0)

Linearity. Fourier Transform of Derivatives

Theorem 1: Linearity

$$\mathfrak{I}(f(x)) = F(w)$$

$$\mathfrak{I}_C(af + bg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-iwx} dx$$

$$\mathfrak{I}(af + bg) = a\mathfrak{I}(f) + b\mathfrak{I}(g)$$

Theorem 2: Fourier transform of the derivative of f(x)

f(x): continuous, f(x) → 0 as |x| → ∞, f'(x) absolutely integrable

$$\mathfrak{I}(f'(x)) = iw\, \mathfrak{I}(f(x))$$

$$\mathfrak{I}(f''(x)) = -w^2\, \mathfrak{I}(f(x))$$

$$\mathfrak{I}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left[fe^{-iwx} \Big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \right]$$

$$\mathfrak{I}(f'') = iw\mathfrak{I}(f') = -i^2 w^2 \mathfrak{I}(f) = w^2 \mathfrak{I}(f)$$

Ex. 3) Fourier transform of $f(x) = xe^{-x^2}$

$$\mathfrak{I}(xe^{-x^2}) = -\frac{1}{2}\mathfrak{I}\left((e^{-x^2})'\right) = -\frac{1}{2} iw \mathfrak{I}\left(e^{-x^2}\right) = -\frac{iw}{2\sqrt{2}} e^{-w^2/4}$$

from *Ex. 2*

Convolution

- Convolution of f * g: $h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p) g(x-p) dp = \int_{-\infty}^{\infty} f(x-p) g(p) dp$

Theorem 3: **Convolution theorem**

$$\Im(f * g) = \sqrt{2\pi}\, \Im(f) \Im(g)$$

$$\Im(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{-\infty} f(p) g(x-p) e^{-iwx} dp\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{-\infty} f(p) g(q) e^{-iw(p+q)} dq\, dp$$

Interchange of integration order
x-p=q

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)^{-iwp} dp \int_{-\infty}^{-\infty} g(q) e^{-iwq} dp = \sqrt{2\pi} \Im(f) \Im(g)$$

Inverse Fourier transform: $(f * g)(x) = \int_{-\infty}^{\infty} F(w) G(w) e^{iwx} dw$ *(see 11.6)*