Modeling, Computers, and Error Analysis

Mathematical Modeling and Engineering Problem-Solving

A Simple Mathematical Model

수학적 모델이라는 것은 수학적 용어로서 물리적 자연현상의 중요한 부분을 식으로서 구성하는 것이다.

![Diagram](image)

*Figure 1.1: The engineering problem-solving process.*
Computers and Software

The Software Development Process

- Programming Style
- Modular Design: Divide into small subprograms
- Top-down Design: Systematic development process
- Structured Programming: How the actual program code is developed

Algorithm Design

- Flowschart: a visual or graphical representation of an algorithm
- Pseudocode: bridges the gap between flowcharts and computer code

Program Composition

- High-level and Macro Languages: C, Fortran, Basic
- Structured Programming
  - consist of the three fundamental control structures of sequence, selection, and repetition
  - only one entrance and one exit
  - Unconditional transfers should be avoided
  - identified with comments and visual devices such as indentation, blank lines, and blank spaces

Quality Control

- Errors or "Bugs"
  - Syntax errors
  - Link or build errors
  - Run-time errors
  - Logic errors
- Debugging
- Testing

Approximations and Round-Off Errors

Significant Figures

Significant figure: The reliability of a numerical value

Accuracy and Precision

- Accuracy: How closely a computed or measured value agrees with the true value
- Precision: How closely individual computed or measured values agree with each other

Error Definitions

- Truncation error: approximations are used to represent exact mathematical procedures
- Round-off error: numbers having limited significant figures are used to represent exact numbers

See Figure 3.10, 3.11 and 3.12 in the textbook.

Truncation Errors and the Taylor Series
The Taylor Series

If a function $f(x)$ can be represent by a power series on the interval $(-a, a)$ then the function has derivative of all orders on that
interval and the power series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \quad (1.1)$$

and this power-series expansion of $f(x)$ about the origin is called a Maclaurin series.

If the expansion is about the point $x = a$, we have the Taylor series

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(0)}{3!}(x - a)^3 + \ldots \quad (1.2)$$

Taylor series specifies the value of function at one point, $x$, in terms of the value of the function and its derivatives at a reference point, $a$. It
is occasionally useful to express a Taylor series in a notation that show how the function behaves at a distance $h$ from a fixed point $a$. If we
call $x = a + h$ in the preceding series, so that $x - a = h$, we get

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(0)}{3!}h^3 + \ldots \quad (1.3)$$

Or with the substitution $a + h \to x_{i+1}$ and $a \to x_i$ we have an alternate form

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \ldots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n \quad (1.4)$$

$R_n$ term is a reminder term to account for all terms from $n + 1$ to infinity:

$$R_n = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x_{i+1} - x_i)^{n+1} \quad (1.5)$$
Mean-value theorem:
If a function $f(x)$ and its first derivative are continuous over an interval from $x_i$ and $x_{i+1}$, then there exists at least one point on the function that has a slope, designated by $f'(\xi)$, that is parallel to the line joining $f(x_i)$ and $f(x_{i+1})$.

See Figure 4.3 in the textbook.

**Using the Taylor Series to Estimate Truncation Errors**

Taylor series expansion of $v(t)$:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + \frac{v''(t_\xi)}{2!}(t_{i+1} - t_i)^2 + \cdots + R_n$$  \hspace{1cm} (1.6)

Truncate the series after the first derivative term:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$  \hspace{1cm} (1.7)

And

$$v'(t_i) = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} - \frac{R_1}{t_{i+1} - t_i}$$  \hspace{1cm} (1.8)
Truncation error is

\[
\frac{R_1}{t_{i+1} - t_i} = \frac{v''(\xi)}{2!} (t_{i+1} - t_i) \quad (1.9)
\]

or

\[
\frac{R_1}{t_{i+1} - t_i} = O(t_{i+1} - t_i) \quad (1.10)
\]

The error of our derivative approximation should be proportional to the step size. Consequently, if we halve the step size, we would expect to halve the error of the derivative.

**Numerical Differentiation**

- **Forward Difference Approximation**
  
  \[
f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(t_{i+1} - t_i)
\]

  or

  \[
f'(x_i) = \frac{\Delta f_i}{h} + O(h)
\]

  where \(\Delta f_i\) is the first forward difference.

- **Backward Difference Approximation**

  \[
f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(t_i - t_{i-1})
\]

  or

  \[
f'(x_i) = \frac{\nabla f_i}{h} + O(h)
\]

- **Central Difference Approximation**

  \[
f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)
\]

Notice that the truncation error is of the order of \(h^2\) in contrast to the forward and backward approximations that were of the order of \(h\). Consequently, the Taylor series analysis yields the practical information that the centered difference is a more accurate representation of the derivative.