System Modeling and Identification

CHBE 702
Korea University
Prof. Dae Ryook Yang
Course Description

• **Emphases**
  – Delivering concepts and Practice
  – Programming Identification Methods using Matlab

• **Class web site**

• **Textbook**

• **References**
System Modeling and Identification

Lecture Note #1
(Chap.1 – Chap.3)

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Chap. 1 Introduction

• **Decision making/Problem Solving**
  - Dependent on access to adequate information about the problem to be solved

• **Form of available information**
  - Data or observations
  - Interpretation is required for further analysis

• **System identification**
  - The derivation of a relevant system description from observed data

• **Model**
  - The resultant system description from system identification
• **Classification of Models**
  - Qualitative/Quantitative
  - Time domain/Frequency domain models
  - Deterministic/Stochastic models
  - SISO/MIMO models
  - Continuous/Discrete-time models
  - Static/Dynamic
  - Black-box/Gray-box/White-box models
  - Parametric/Nonparametric models
  - Linear/Nonlinear models

• **Purposes of the model**
  - Prediction: future system behavior
  - Learning new rules: account for different situations
  - Data compression: compact form and low complexity
• **Procedure of Identification**
  
  - A System to study
  - Purpose and problem formulation
  - Experimental planning and operation to ensure that the prerequisites of the identification methods are used in the experimental procedure
  - A model set
  - Identification and parameter estimation methods
  - Model validation
Basics of Statistics

• Basic knowledge on statistics
  – Choose any undergraduate level statistics course.
  – Get to know the basic concepts.
  – Other materials will be provided.
Chap. 2 Black Box Models

- **Black box models**
  
  - Impulse response model, $g(t)$
    
    $$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

    For impulse input, $u(t) = \delta(t) \rightarrow y(t) = g(t)$

  - Frequency response model
    
    $$Y(s) = G(s)U(s)$$

    where $Y(s) = \mathcal{L}\{y(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$, $U(s) = \mathcal{L}\{u(t)\}$
Difficulties in Time Domain Analysis (1)

- Impulse response analysis

\[ g_1(t) = 1.05 \cdot 0.41' \]
\[ g_2(t) = 1.2 \cdot 0.5' - 0.2 \cdot 0.75' \]

- FIR’s look very close.
- FSR’s are different
- Low freq. information (gain) is hard to get.
Difficulties in Time Domain Analysis (2)

• Impulse response analysis

\[ g_1(t) = 0.7^t - 0.1^t \]
\[ g_2(t) = 0.8 \cdot 0.73^t \]

- At the sampling times, FIR’s look very close.
- FSR’s are quite different.
Practical Problems of Impulse Response Analysis

- Restriction to stable systems
- Difficulties in generating impulses
- Dynamic sample and hold elements
- Synchronization between impulse and sampling
- Difficulties for the system in managing inputs of large magnitude
- Saturations
- Nonlinearities
- Difficulties in handling the tails of responses due to their long duration and low amplitudes
- Sensitivity to noise
Difficulties in Time Domain Analysis (3)

• Step response analysis

\[ u(t) = \begin{cases} 
1, & t > 0 \\
0, & t \leq 0 
\end{cases} \quad y(t) = \int_{0}^{t} g(\tau)d\tau \]

– A good estimate of static gain can be obtained.
– FIR can be obtained by differentiating step response.

\[ \hat{g}(t) = \frac{d}{dt} y(t) \]

– The noise will affect the accuracy of the model.
– Proper filtering will improve the accuracy.
Frequency Response Analysis

- For sinusoidal input, \( u(t) = u_1 \sin \omega t \),
  \[
y(t) = |G(i\omega)| u_1 \sin(\omega t + \phi(\omega)); \quad \phi(\omega) = \arg G(i\omega)
  \]

- Signal probing
  \[
s_T(\omega) = \int_0^T y(t) \sin \omega t \, dt = \frac{1}{2} T |G(i\omega)| u_1 \cos \phi(\omega); \quad T = kh = \frac{2\pi}{\omega} k
  \]
  \[
c_T(\omega) = \int_0^T y(t) \cos \omega t \, dt = \frac{1}{2} T |G(i\omega)| u_1 \sin \phi(\omega)
  \]
  \[
  \left| \hat{G}(i\omega) \right| = \frac{2}{Tu_1} \sqrt{s_T^2(\omega) + c_T^2(\omega)}
  \]
  \[
  \hat{\phi}(i\omega) = \arctan \frac{c_T(\omega)}{s_T(\omega)} + k\pi
  \]
Difficulties in Frequency Domain Analysis (1)

- Process with sinusoidal disturbance

\[ Y(s) = G(s)U(s) + V(s) \]

- The disturbance \( v(t) \) is with period 10Hz (62.8 rad/s).
- If the test is corrupted by noise, the frequency response may be modified significantly by noise.
- The disturbance affects both the gain and phase over a large freq interval.
Difficulties in Frequency Domain Analysis (2)

- **Sampling interference**

  \[ G(s) = \frac{1}{s+1} \]

  - From the input/output data, the discrete-time method can generate the frequency response. (FFT, DFT, etc.)
  - If the sampling frequency is 10Hz, the frequency response has significant error in the freq range above the 1Hz.
Practical Problems of Frequency Response Analysis

- Disturbances acting on output
- Sampling interference
- Unmodeled nonlinearities
- Presence of higher-order harmonics
- Only the sinusoidal input can be applied
- One experiment is needed for each test frequency
- Long measurement time is required
- Restricted to stable systems
- Restricted to time-invariant systems
- Transient response should be discarded
Chap. 3 Signals and Systems

- **Laplace transform**
  \[ X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt; \quad s = \sigma + i\omega \text{ (complex freq.)} \]
  - Valuable for analysis of transient behavior
  - One-sided LT: \[ X(s) = \mathcal{L}[x(t)] = \int_{0}^{\infty} x(t)e^{-st} \, dt \]

- **Fourier transform (spectrum of } x(t))**
  \[ X(i\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-i\omega t} \, dt; \]
  - Valuable for periodic signals
  - LT and FT coincide for the choice \[ s = i\omega. \]
Discretized Data

- **Sampled signal**
  - Sampling period: \( h \)
  - Sample sequence: \( \{ x_k \}_{-\infty}^{\infty} ; \ x_k = x(kh) \) for \( k = \ldots, -1, 0, 1, 2, \ldots \)
  - Sampled function The average of \( x(t) \) and \( x_\Delta(t) \) to be same
  \[
  x_\Delta(t) = x(t)h \sum_{k=-\infty}^{\infty} \delta(t - kh) = x(t)W_h(t); \ W_h(t) \triangleq h \sum_{k=-\infty}^{\infty} \delta(t - kh)
  \]
  - Spectrum of sampled signal
    \[
    X_\Delta(i\omega) = \mathfrak{F}[x_\Delta(t)] = \mathfrak{F}[x(t)] \otimes \mathfrak{F}[W_k(t)] \quad (\otimes \text{ is convolution})
    \]
    \[
    X_\Delta(i\omega) = \sum_{k=-\infty}^{\infty} X \left( i(\omega - 2\pi k / h) \right) \quad \text{since} \quad \mathfrak{F}[W_k(t)] = (h / 2\pi)W_{2\pi/h}(\omega)
    \]
    \( (X_\Delta(i\omega) \) is a periodic function of the original spectrum \( X(i\omega) \) )
The Shannon’s Sampling Theorem

- The continuous-time variable $x(t)$ may be reconstructed from the samples if and only if the sampling frequency is at least twice that of the highest frequency for which $X(i\omega)$ is non zero.

- The sampling frequency: $\omega_s$
- The Nyquist frequency: $\omega_n = \omega_s / 2$
- The Nyquist frequency indicates the upper limit of distortion-free sampling.

A variable cannot be sampled in a finite measurement interval without spectral distortion arising. (spectral leakage)
The Discrete-time Transforms

- **Definition of Z-transform**
  \[ X_z(z) = Z[x(t)] = \sum_{k=-\infty}^{\infty} x_k z^{-k} \]

- Z-transform exists only if \( \sum_{k=-\infty}^{\infty} |x_k||z^{-k}| < \infty \)

- \( X_{\Delta}(i\omega) = \mathcal{F}[x(t)W_h(t)] = h \sum_{k=-\infty}^{\infty} x_k \exp(-i\omega kh) = hX_z(e^{i\omega h}) \)

- **Discrete Fourier transform (DFT)**
  - \( N \) measurements with sampling time \( h \)
  \[ X_k = \mathcal{F}_{\Delta(h,N)}[x(kh)] = h \sum_{l=0}^{N-1} x_l \exp(-i\omega_k lh) = hX_z(e^{i\omega_k h}) \]

where \( \omega_k = (2\pi / Nh)k \) for \( k = 0, 2, \cdots, N - 1 \)
Signal power and Energy

- Instantaneous power of signal $x$ at time $t$
  $$p_{xx}(t) = x(t) \cdot x^*(t) \quad (* \text{ is complex conjugate transpose})$$

- Instantaneous power of interaction between $x$ and $y$
  $$p_{xy}(t) = x(t) \cdot y^*(t) = p_{yx}^*(t)$$

- Energy of a signal
  $$e_{xx} = \int_{-\infty}^{+\infty} x(t) \cdot x^*(t) dt$$

- Interaction energy between $x$ and $y$
  $$e_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t) \cdot y^*(t-\tau) dt = e_{yx}^*(-\tau)$$

- Cross covariance
  $$C_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) \cdot y^*(t-\tau) dt$$
Spectra and Covariance Functions

- Spectral density (energy spectrum)
  \[ E_{xx}(i\omega) = X(i\omega) \cdot X^*(i\omega) \]

- Cross energy spectrum between \( x \) and \( y \)
  \[ E_{xy}(i\omega) = X(i\omega) \cdot Y^*(i\omega) \]
  - Parseval relations (Signal energies are same in both domains)
    \[ \int_{-\infty}^{+\infty} x(t) \cdot y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(i\omega) \cdot Y^*(i\omega) d\omega \]

- Autospectrum
  \[ S_{xx}(i\omega) = \mathcal{F} \left[ C_{xx}(\tau) \right] \]

- Power cross spectrum
  \[ S_{xy}(i\omega) = \mathcal{F} \left[ C_{xy}(\tau) \right] \]
• Power cross spectra from linear systems

  - The process: \( y(t) = x(t) + v(t) = g(t) * u(t) + v(t) \)

  \[
  S_{yy}(i\omega) = G(i\omega)S_{uu}(i\omega)
  \]

  \[
  S_{uy}(i\omega) = S_{uu}(i\omega)G^*(i\omega)
  \]

  \[
  S_{yy}(i\omega) = G(i\omega)S_{uu}(i\omega)G^*(i\omega) + S_{vv}(i\omega)
  \]

  \[ e_{xy} = 0 \]

  - The signal-to-noise ratio (SNR): \( x \) and \( v \) are uncorrelated

    \[
    \text{SNR} = \frac{e_{xx}}{e_{vv}} = \frac{e_{yy}}{e_{vv}} - 1
    \]

  - Correlation coefficient/Quadratic coherence spectrum

    \[
    \rho(\tau) = \frac{C_{xy}(\tau)}{\sqrt{|C_{xx}(\tau)|} \sqrt{|C_{yy}(\tau)|}}
    \]

    \[
    \gamma_{xy}^2(\omega) = \frac{|S_{xy}(i\omega)|^2}{S_{xx}(i\omega)S_{yy}(i\omega)}
    \]
• Coherent function expresses the degree of linear correlation in the frequency domain between the input and output

\[
\gamma_{uy}^2(\omega) = \frac{|S_{uy}(i\omega)|^2}{S_{uu}(i\omega)S_{yy}(i\omega)} = \frac{|G(i\omega)|^2 S_{uu}^2(i\omega)}{S_{uu}(i\omega)(|G(i\omega)|^2 S_{uu}(i\omega) + S_{vv}(i\omega))}
\]

\[
= \frac{1}{1 + \frac{S_{vv}(i\omega)}{S_{uu}(i\omega)|G(i\omega)|^2}}
\]

– If coherent function is close to unity, it implies that
  • the noise level is low and
  • there is a linear response of \(y(t)=g(t)\ast u(t)+v(t)\) between input and output.
Random variable (stochastic variable), $X$
- Has a value which is independent on chance
- Cannot be predicted from a knowledge of the experimental conditions

Probability that $X \leq x$
$$F(x) = P\{X \leq x\}, \quad 0 \leq F(x) \leq 1, \quad \forall x \in R$$

Probability density function
$$f(x) = \frac{dF(x)}{dx}$$

$\alpha$-percentile of the distribution
$$P\{X \leq x_\alpha\} = \alpha$$

The mean (expectation) of the distribution
$$\mu_x = E\{x\} = \int_{-\infty}^{\infty} xf(x)dx$$
• **The variance**

\[ \text{Var}\{x\} = E\{(x - \mu_x)(x - \mu_x)^T\} = \int_{-\infty}^{\infty} (x - \mu_x)(x - \mu_x)^T f(x)dx = \sigma_x^2 \]

• **The mean of variable** \( y \) **which is a function of** \( X \)

\[ \mu_y = E\{y\} = \int_{-\infty}^{\infty} y(x) f(x)dx \]

• **The covariance between** \( x \) **and** \( y \)

\[ \text{Cov}\{x, y\} = E\{(x - \mu_x)(y - \mu_y)^T\} = \int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y)^T f(x)dx \]

\[ = E\{xy^T\} - \mu_x\mu_y^T \]

• **Statistically independent variables**

\[ f(x, y) = f_1(x)f_2(y) \]

• **Statistical covariance and correlation coefficient**

\[ \rho = \text{Cov}\{x, y\} / (\sigma_x \sigma_y) \quad (x \text{ and } y \text{ are uncorrelated if } \text{Cov}\{x, y\} = 0) \]

• **The** \( p \)-**th moment**

\[ E\{x^p\} = \int_{-\infty}^{\infty} x^p f(x)dx \]
• **The normal or Gaussian distribution for** $x \in \mathbb{R}^N$

$$f_x(x) = \frac{1}{(2\pi)^{N/2}(\text{det } R)^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu_x)^T R^{-1} (x - \mu_x) \right\}$$

$$\mu_x = E\{x\} = \begin{pmatrix} E\{x_1\} \\ \vdots \\ E\{x_N\} \end{pmatrix} \quad R = \text{Cov}\{x, x\} = \begin{pmatrix} \text{Cov}\{x_1, x_1\} & \cdots & \text{Cov}\{x_1, x_N\} \\ \vdots & \ddots & \vdots \\ \text{Cov}\{x_N, x_1\} & \cdots & \text{Cov}\{x_N, x_N\} \end{pmatrix}$$

• **Linear transformations**
  
  – For $y = Ax + b$ ($x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$)

$$E\{y\} = AE\{x\} + b = A\mu_x + b$$

$$\text{Cov}\{y, y\} = E\{(Ax + b - (A\mu_x + b))(Ax + b - (A\mu_x + b))^T\} + b$$

$$= ACov\{x, x\}A^T \geq 0$$
• **Time series**
  - A function \( x(t) = x(t,w) \) whose values depend on a random variable \( w \) is called a **random or stochastic process**. (This function is also a r.v.)
  - For a fixed \( w \), \( x(t,w) \) is only a function of \( t \) and is called a **realization of the stochastic process** or **sample function**.
  - In discrete time, infinite sequence of \( \{x_k\} \ (k=1,\ldots,\infty) \) or a sequence \( \{x_k\} \ (k=1,\ldots,N) \) over some interval of time is called **time series**.

• **White noise**
  - A sequence of \( N \) uncorrelated stochastic variable \( \{w_i\} \ (k=1,\ldots,N) \) with \( \text{E}\{w_i\}=0 \), \( \text{E}\{w_i,w_j\}=\delta_{ij}\sigma^2 \) for all \( i,j \) is known as **white noise** in the domain of time-series analysis.

• **Stationary Processes**
  - A random process \( x(t,w) \) is called **weakly stationary** if its expectation \( \mu_x(t)=\text{E}\{x(t,w)\} \) is constant and independent of time \( t \) and if the covariance functions \( C_{xx}\{t_1, t_2\}=\text{Cov}\{x(t_1), x(t_2)\} \) depends only on time shift \( \tau=t_1-t_2 \).
  - A stochastic process \( x(t,w) \) is called **strictly stationary** if the joint probability distribution of some set of \( N \) observations \( x_1, x_2, \ldots, x_N \) is the same as that associated with the \( N \) observations \( x_1+k, x_2+k, \ldots, x_N+k \) for any \( k \).
  - Two stochastic processes \( \{x_k\} \) and \( \{y_k\} \) are uncorrelated if and only if cross covariance function \( C_{xy}\{\tau\}=0 \) for all \( \tau \).